FORMULATION OF SEQUENCE OF INTEGER SOLUTIONS TO THE CONE

\[ z^2 = 34x^2 + y^2 \]

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Abstract:

This paper aims at determining a formula for generating sequence of integer solutions to the cone represented by the ternary quadratic diophantine equation given by \( z^2 = 34x^2 + y^2 \).

Introduction:

Consider the cone represented by

\[ z^2 = 34x^2 + y^2 \]  \( (1) \)

Expressing (1) as a system of double equations and performing a few calculation, it is seen that (1) is satisfied by the triples \( (2k, 33k, 35k), (2rs, 34r^2 - s^2, 34r^2 + s^2) \) and \( (2k, 2k^2 - 17, 2k^2 + 17) \).

Thus, it is observed that (1) has an infinite number of integer solutions. However, the main thrust of this paper is to obtain a formula for generating sequence of integer solution to the given cone based on its given solution. In this context, one may refer \([1-8]\).

Method of analysis:

Let \((x_0, y_0, z_0)\) be any solution of (1).

The solution may be in real integers or in gaussian integers or in irrational numbers.

Formula: 1

Let \((x_1, y_1, z_1)\) be the second solution of (1), where

\[ x_1 = h + x_0, \ y_1 = y_0 + h, \ z_1 = 6h - z_0 \]  \( (2) \)

in which h is an unknown to be determined.

Substitution of (2) in (1) gives

\[ h = 68x_0 + 2y_0 + 12z_0 \]  \( (3) \)

Using (3) in (2), the second solution \((x_1, y_1, z_1)\) of (1) is expressed in the matrix form as
\[(x_1, y_1, z_1) = M(x_0, y_0, z_0)\]

where \(t\) is the transpose and

\[
M = \begin{pmatrix}
69 & 2 & 12 \\
68 & 3 & 12 \\
408 & 12 & 71
\end{pmatrix}
\]

The repetition of the above process leads to the general solution \((x_{n+1}, y_{n+1}, z_{n+1})\) of (1) written in the matrix form as

\[
\begin{pmatrix}
x_{n+1} \\
y_{n+1} \\
z_{n+1}
\end{pmatrix} = \begin{pmatrix}
\frac{34Y_n + 1}{35} & \frac{Y_n - 1}{35} & X_n \\
34(Y_n - 1) & Y_n + 34 & X_n \\
34X_n & X_n & Y_n
\end{pmatrix} \begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}, \quad n = 0, 1, 2, \ldots
\]

where \((X_n, Y_n)\) is the general solution of the Pellian equation \(Y^2 = 35X^2 + 1\), \(X_0 = 12, Y_0 = 71\).

**Formula: 2**

Let \((x_1, y_1, z_1)\) be the second solution of (1), where

\[
x_1 = x_0 + h, \quad y_1 = y_0, \quad z_1 = 6h - z_0
\]

(4)

Substituting (4) in (1) and performing a few calculations, it is seen that

\[h = 34x_0 + 6z_0\]

and thus, \(x_1 = 35x_0 + 6z_0\); \(z_1 = 204x_0 + 35z_0\)

which is written in the matrix form as

\[
\begin{pmatrix}
x_1 \\
z_1
\end{pmatrix} = M \begin{pmatrix}
x_0 \\
z_0
\end{pmatrix}
\]

where \(M = \begin{pmatrix} 35 & 6 \\ 204 & 35 \end{pmatrix} \)
Proceeding in a similar manner, we have, in general

\[
\begin{pmatrix}
  x_n \\
  z_n
\end{pmatrix} = M^n \begin{pmatrix}
  x_0 \\
  z_0
\end{pmatrix}
\]

where

\[
M^n = \frac{1}{2\sqrt{34}} \begin{pmatrix}
  \sqrt{34}(\alpha^n + \beta^n) & (\alpha^n - \beta^n) \\
  34(\alpha^n - \beta^n) & \sqrt{34}(\alpha^n + \beta^n)
\end{pmatrix}
\]

in which \( \alpha = 35 + 6\sqrt{34} \), \( \beta = 35 - 6\sqrt{34} \)

Thus, the general formula for obtaining a sequence of non-zero distinct integer solutions based on the given solution to (1) is represented by

\[
\begin{aligned}
  x_n &= \frac{1}{2\sqrt{34}} \left[ \sqrt{34}(\alpha^n + \beta^n)x_0 + (\alpha^n - \beta^n)z_0 \right] \\
  y_n &= y_0 \\
  z_n &= \frac{1}{2\sqrt{34}} \left[ 34(\alpha^n - \beta^n)x_0 + \sqrt{34}(\alpha^n + \beta^n)z_0 \right]
\end{aligned}
\]

**Formula: 3**

Let \( x_1 = 3x_0, y_1 = 3y_0 + h, z_1 = 2h - 3z_0 \) be the second solution of (1). Repeating the above process, the corresponding general formula representing the solutions to (1) is

\[
\begin{aligned}
  x_n &= 3^n x_0 \\
  y_n &= \frac{1}{2} \left[ (\alpha^n + \beta^n)y_0 + (\alpha^n - \beta^n)z_0 \right] \\
  z_n &= \frac{1}{2} \left[ (\alpha^n - \beta^n)y_0 + (\alpha^n + \beta^n)z_0 \right]
\end{aligned}
\]

**Formula: 4**

Let \( x_1 = h - 35x_0, y_1 = h - 35y_0, z_1 = 35z_0 \) be the second solutions of (1). Repeating the above process, the corresponding general formula representing the solutions to (1) is

\[
\begin{aligned}
  x_n &= \frac{1}{35} \left[ (34\alpha^n + \beta^n)x_0 + (\alpha^n - \beta^n)y_0 \right] \\
  y_n &= \frac{1}{35} \left[ 34(\alpha^n - \beta^n)x_0 + (\alpha^n + 34\beta^n)y_0 \right] \\
  z_n &= 35^n z_0
\end{aligned}
\]

References:


